Price war or tacit collusion:
A repeated game approach to oligopoly

Patrick Maillé, Maurizio Naldi, Bruno Tuffin

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A very simple oligopoly model

Take a simple example:

- $N = 2$ identical providers in competition, with (fixed) QoS for each provider i
- Demand model: *separable* demand function   Allon&Federgruen'08

$$D_i := D_0 - b_i p_i + \sum_{j \neq i} \beta_{ij} p_j, \quad i = 1, ..., N$$

  with p_i the price charged by provider i, and b_i, β_{ij} positive constants. Here we assume $b_i = b$ and $\beta_{ij} = \beta < 2b$ for all i, j.

Thus here

$$\begin{aligned} D_1 &= D_0 - bp_1 + \beta p_2 \\ D_2 &= D_0 - bp_2 + \beta p_1 \end{aligned}$$

Then look at the Nash equilibrium of the competition game (where utility=revenue):

$$U_1(p_1, p_2) = p_1 D_1 = p_1 \left(D_0 - bp_1 + \beta p_2\right),$$

maximum for $p_1 = (D_0 + \beta p_2)/2b$ ⇒symmetric equilibrium:

$$p_1^N = p_2^N = p^N = D_0/(2b - \beta)$$

$$U_1^N = U_2^N = U^N = b\left(\frac{D_0}{2b-\beta}\right)^2$$

On the other hand, if both providers cooperate to maximize $U_1 + U_2$, we get $U_i(p,p) = p\left(D_0 - (b-\beta)p\right)$, that is maximum for $p^* = \frac{D_0}{2b-2\beta} > p^N$, and then

$$U_1^* = U_2^* = U^* = \frac{D_0^2}{4(b-\beta)} = U^N \frac{(2b-\beta)^2}{4b(b-\beta)} > U^N$$

⇒ by cooperating, providers set higher prices and both increase their revenue.

Provider competition as a Prisoner's dilemma

- Each provider has an interest in lowering his price to attract clients
- If both do, the price war leads to a revenue reduction.
- To fix ideas, assume only the strategies p^N and p^* are available, and set $D_0 = 3, b = 2, \beta = 1$, so that the game is

	(price war) $p^N = 1$	(cooperation) $p^* = 3/2$
(price war) $p^N = 1$	$(2, 2)$	$(5/2, 3/2)$
(cooperation) $p^* = 3/2$	$(3/2, 5/2)$	$(9/4, 9/4)$

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Only one **Nash equilibrium**, but another solution **Pareto-dominates** it.

Joint-maximization and repeated games

	$p^N = 1$	$p^* = 3/2$
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- Providers may have interest in cooperating so as to maximize the sum of their benefits Chamberlin 1933
- But the temptation to cheat is very strong
- Franklin M. Fisher (MIT): Fisher 1989

  The study of any real oligopoly has largely become the study of how the joint-optimization solution is or is not achieved and the reasons why.[...]
  Tacit collusion is only made possible by the fact that the game, or games like it, will be played again.

Competition **is** a repeated game

- Consider the time dimension: the game is not played only once but for several *periods* (months, years).
- Most classical model: at each period the *same game* is played (not necessarily true here, but maybe a good approximation if churning is easy)
- Most common model: players (providers) try to maximize the utility (weighted average payoff):

$$A_i := (1-\delta)\sum_{t=0}^{\infty}\delta^t U_i(a_t), \quad \text{where}$$

  - $\delta \in (0,1)$ is the *discount factor*. Interpretations:
    $\begin{cases} \delta = \frac{1}{1+\underbrace{r}_{\text{interest rate}}}, \\ \delta = \mathbb{P}(\text{game continues at next period}), \end{cases}$ or a combination of both;
  - a_t is the vector of players actions at period t;
  - U_i is the corresponding utility for player i.

Main results on repeated games (1/3)

We denote by $U_{i,m}$ the minimax utility of player i, i.e.

$$\begin{aligned} U_i^{\min} &:= \min_{a_{-i}} \max_{a_i} U_i(a_i, a_{-i}) \\ &= \text{the worse utility that } i \text{ can ensure if he knows } a_{-i} \\ &= \text{the worse utility that the others can impose to player } i. \end{aligned}$$

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Theorem (The Folk Theorem)

For any $(v_1, ..., v_N)$ in the convex hull of the "attainable" utility vectors, such that $v_i > U_i^{\min}$, if δ is sufficiently close to 1 then there exists a Nash equilibrium of the infinitely repeated game where, for all i, $A_i = v_i$.

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Proof.

The utility vector $(v_1, ..., v_N)$ is reachable via a (correlated) strategy vector $(s_1, ..., s_N)$.
Very simple Nash strategies: each player i plays s_i while everybody does, and if a player j deviates then i plays j's minimax sanction forever. □

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Theorem (Aumann-Shapley, 1976/Rubinstein, 1979)

If $\delta = 1$, then the Folk Theorem holds with **perfect** *Nash equilibria.*

Proof: players should sanction those deviate, those who did not sanction, and so on.

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Theorem (Friedman, 1977)

If $v_i > U_i(a^N)$ where a^N is a (one-shot) Nash equilibrium, then if δ is sufficiently close to 1, there exists a **perfect** *Nash equilibrium of the infinitely repeated game where, for all i, $A_i = v_i$.*

Simple proof: player i plays s_i until someone deviates, then plays a_i forever.

Main results on repeated games (3/3)

Theorem (Fudenberg&Maskin, 1986)

Assume that the convex hull V^* of the reachable utility vectors dominating the minimax utilities is of dimensionality N, then for any $(v_1, ..., v_N) \in V^*$, if δ is sufficiently close to 1 then there exists a subgame-perfect equilibrium of the infinitely repeated game in which player i's average payoff is v_i.

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Proof.

Simple principle: if one player deviates, he is “minimaxed” by the other players long enough for the sanction to overcome the gain from the deviation, and afterwards sanctioners get “rewarded” for having sanctioned. □

Consequences for our problem

- Competition is supposed to lead to price reductions that benefit to the users
- We would like providers to play the (one-shot) Nash equilibrium (at least for our example)

BUT

- If δ is sufficiently large then almost anything (in particular, collusion) can be the outcome of the repeated game.
- **Idea**: "reduce" δ through regulation

Regulation rule: price stability

Provider can change actions (prices) only every k periods: for each user i and each $m \in \mathbb{N}$ we impose $a_{km+l} = a_{km}$ if $l < k$.
Now the objective of each player is to maximize

$$
\begin{aligned}
A_i &= (1-\delta)\sum_{t=0}^{\infty}\delta^t U_i(a_t) \\
&= (1-\delta)\sum_{m=0}^{\infty}\delta^{km}\left(U_i(a_{km}) + \delta U_i(a_{km+1}) + \ldots + \delta^{k-1}U_i(a_{km+k-1})\right) \\
&= (1-\delta)(\sum_{l=0}^{k-1}\delta^l)\sum_{m=0}^{\infty}(\delta^k)^m U_i(a_{km}) \\
&= (1-\delta^k)\sum_{m=0}^{\infty}(\delta^k)^m U_i(a_{km})
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So we changed δ into δ^k!!

Back to our example

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Here, joint-maximization can be attained by the perfect Nash strategy for each provider :

- Play p^* while the other does,
- If the other deviates, then play p^N forever.

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This is a Nash equilibrium if the sanction in future periods $((9-8) \times \sum_{m=1}^{\infty} \delta^m = \delta/(1-\delta))$ exceeds the immediate gain from deviating $(10 - 9 = 1)$, i.e. if $\delta/(1-\delta) > 1$, or $\delta > 1/2$.

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On the contrary, if $\delta < 1/2$ then no player can prevent the other from playing p^N instead of p^*.
$\Rightarrow$ if we choose a "price stability period" k such that $\delta^k < 1/2$ then no collusion should occur *in that game*.

Things are a bit more complicated...

- Not only two strategies, but a continuum
- It would be interesting to see the influence of k
  - on the reachable outcomes (via Nash or perfect Nash equilibria)
  - on the "value of collusion": what can be earned at most by a (Nash sustainable) collusion outcome.

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Other possible interesting things to do

- If providers are not identical, do we need stronger regulation on the ones with significant market power?
- Look at $N > 2$
- Introduce some QoS term in the demand Allon&Federgruen'08